V-Variable Totally Disconnected *n*-gonal Superfractals Using Ishikawa Iterates

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Abstract: The concept of totally disconnected IFS (Iterated Function System) and their as- sociated disconnected fractal sets have basically been given by Barnsley [1, 2]. Many researchers have developed n-gonal fractals using simple function iterates. The aim of this paper is to develop V - variable pentagon and hexagon as a super fractal of totally disconnected hyperbolic IFS using the Ishikawa iterative procedure. A general algorithm to find vertices and scaling factor of a regular n-gon and generation of n-gon is also given, wherein each vertex of n-gon is equivalent to some root of a chosen number.

Keywords: Disconnected IFS, V-variable superfractals, n-gon fractal, Ishikawa iterates.

I. INTRODUCTION

Hutchinson [12] introduced a backward-iteration method for constructing deterministic fractal sets. Later, Barnsley and Demko [3] proposed an alternative framework based on forward iteration. Random fractal sets were subsequently obtained by Falconer [9], Graf [10], and Mauldin and Williams [15], who incorporated randomness into each backward step. The forward scheme, on the other hand, makes it possible to efficiently generate accurate approximations of stochastic processes. An interesting feature of V-variable random fractals is that they can also be produced through a forward construction. Here, the parameter V represents the level of allowable variation across scales, and a V-variable fractal exhibits no more than V distinct local forms.

The concept of totally disconnected IFS is nicely discussed and their associated disconnected fractal sets are given by Barnsley [1, 2]. The V-variable fractals have been discussed by Singh et al. [21] and Rani et al. [16], which are helpful in modeling various natural objects with the help of a superior iterative procedure. Indeed, V-variable fractals are the bridge between deterministic fractals and random fractals. The development of regular n-gons is given by Riddle [19]. A method to develop the vertices of a regular n-gon is described by Churchill and Brown [6].

There are basically three types of IFSs: a totally disconnected IFS, an overlapping IFS and a just touching IFS [1, 2]. Rani et al. [16] discussed just touching IFS and associated V-variable Sierpiński carpet and Sierpiński gaskets.

The development of totally disconnected V-variable pentagons and hexagons is described in this paper along with the algorithm which gives vertices of these n-gons along with their scaling factor. This paper provides an example of a totally disconnected super IFS as well.

The process of convergence always depends on an IFS and the iterative procedure which we select. In this paper, we use a relatively faster iterative process known as the Ishikawa iterative process to model V -variable disconnected pentagons and hexagons, instead of using Picard iterates. Section 2 discusses basic definitions and background that forms the basis of our paper. Section 3 deals with the development of disconnected hyperbolic IFS for pentagonal and hexagonal superfractals along with the choice of scaling factor. Section 4 ends with concluding remarks.

II. PRELIMINARIES

Hexagons frequently appear in natural structures; for instance, honeycombs built by bees exhibit hexagonal geometry [8, 22]. The construction of n-gonal fractals also plays a crucial role in the development of fractal multiband antennas [7]. Tang [23] discussed the design of such an antenna based on a hexagonal geometry.

Determining vertex locations is essential for generating polygonal fractals and for obtaining the affine mappings that yield the corresponding fractal attractors. Algorithm 1 outlines the process of computing vertices and scaling coefficients. The centre of each polygon constructed is assumed to coincide with the origin. Using De Moivre's theorem, the coordinates of the vertices of an n-gon are obtained via the nth roots of a selected number so that consecutive vertices are exactly one unit apart [6]. All vertices lie at equal distance from the origin. Subsequently, translation and rotation are applied so that the base edge aligns with the segment connecting (0,0) and (1,0). Once these n vertices and the scale parameter are identified, affine mappings can be derived that generate the required n-gonal fractal.

Following [1, 4], we adopt the following terminology and notation.

Definition 1 ([1, p. 316]). Hyperbolic IFS: Let (Y, ρ) be a complete metric space and consider a finite set of strict contractions

$$g_k : Y \to Y, \qquad k = 1, 2, \ldots, N.$$

Then

$$\{Y; g_1, g_2, \ldots, g_N\}$$

is called a contractive iterated function system or hyperbolic IFS.

Super IFS. Let (Y, ρ) be compact. Let M and N be positive integers. For each index n ∈ {1, . . . , N }, define

$$\mathcal{G}^n = \{Y; g_1^n, g_2^n, \ldots, g_M^n ; q_1^n, q_2^n, \ldots, q_M^n \},$$

where each $g_m^n : Y \to Y$ is contractive and the associated probabilities satisfy

$$\sum_{m=1}^{M} q_m^n = 1, \qquad q_m^n \geq 0.$$

Now define

$$\mathcal{G} = \{Y; \mathcal{G}^1, \mathcal{G}^2, \ldots, \mathcal{G}^N ; Q_1, Q_2, \ldots, Q_N\},$$

with

$$\sum_{n=1}^{N} Q_n = 1, \qquad Q_n \geq 0.$$

A totally disconnected hyperbolic IFS yields a disconnected attractor [2]. Let H (Y) denote the set of all nonempty compact subsets of Y. The attractor A is the unique fixed point of the contraction

$$\mathcal{G} : \mathcal{H}(Y) \to \mathcal{H}(Y), \quad \mathcal{G}(B) = g_1(B) \cup g_2(B) \cup \cdots \cup g_N(B),$$

see [4, 11, 12, 15]. Equivalently,

$$A = g_1(A) \cup g_2(A) \cup \cdots \cup g_N(A).$$

For this attractor A, define the set of overlaps

$$O_{\mathcal{G}} = \{ g_i(A) \cap g_i(A) : i \neq j, \ i, j \in \{1, \ldots, N\} \}.$$

We clarify $\mathcal{G}$ by:

- totally disconnected if $O_{\mathcal{G}} = \emptyset$;
- overlapping if $O_{\mathcal{G}}$ contains a nonempty relatively open subset in A;
- just-touching otherwise.

Proposition 1. The Hausdorff dimension of a Sierpiński n-gon is

$$dim_H = \frac{ln\,(n)}{ln\,(\alpha_n)},$$

Where

$$\alpha_n = 2\left(1 + \sum_{k=1}^{\infty} \cos\frac{2k\pi}{n}\right).$$

Remark 1. A scaling ratio $r < 1/\alpha_n$ gives a disconnected IFS; $r > 1/\alpha_n$ results in overlap, while $r = 1/\alpha_n$ corresponds to the just-touching case.

Iterative techniques are essential in constructing fractal systems. We now recall Mann and Ishikawa schemes.

Mann Iterates. Let (Y, ρ) be compact with $Y \subseteq \mathbb{R}$. Given $T : Y \to Y$, the Mann iteration is defined as

$$x_{k+1} = (1 - \theta_k)\, x_k + \theta_k T\,(x_k),$$

where $\theta_k \in [0, 1]$.

Ishikawa Iterates. Let (Y, ρ) be compact with $Y \subseteq \mathbb{R}$. The Ishikawa iteration is

$$\mathcal{Y}_k = (1 - \eta_k)x_k + \eta_k T(x_k),$$

$$x_{k+1} = (1 - \theta_k)x_k + \theta_k T(\mathcal{Y}_k),$$

where $\theta_k, \eta_k \in [0, 1]$.

The Ishikawa iteration reduces to the Mann method if

$$\eta_k = 0, \qquad \forall k,$$

and the Mann iteration reduces to the Picard iteration when

$$\theta_k = 1, \qquad \forall k.$$

Throughout this paper we consider the case

$$\theta_k = \theta \quad \text{and} \quad \eta_k = \eta.$$

III. V- VARIABLE DISCONNECTED n-GONAL SUPERFRACTALS

Using complex variable methods, an algorithm has been formulated to compute the vertices of a polygon with n ≥ 3 vertices. For example, n = 3,4,5,6 correspond to a triangle, square, pentagon and hexagon, respectively. Once the vertices are located and connected by straight segments, an n-gon is obtained, from which an appropriate affine IFS may be selected. Applying the concept of V-variability and iterating via Ishikawa updates yields the desired superfractal. A schematic representation (Fig. 1) and Algorithm 1 describe this procedure, where zk denotes the vertices calculated through De Moivre's theorem (cf. [6]).

Algorithm 1

1. Provide the input n, representing the number of polygon vertices, where n ≥ 3.

2. Compute the vertex locations w_k, where

$$w_k = \text{cis}\left(\frac{2k\pi}{n}\right),\ \kappa = 0,1,2,\ldots,n-1.$$

3. Plot these points wk for k = 0,1,...,n − 1.

4. Connect wk to wk+1 for k = 0,1,2,...,n−1 to obtain an n-sided polygon.

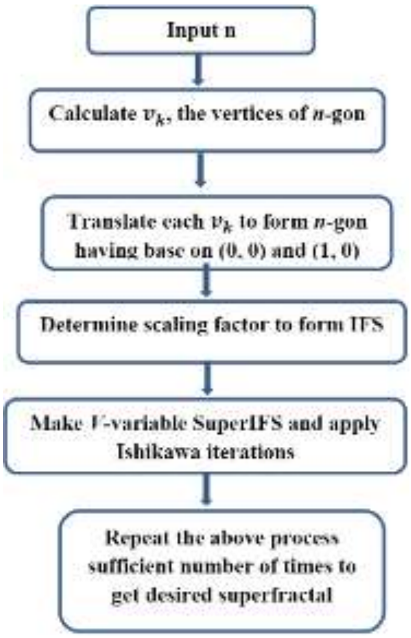

Figure 1: A descriptive caption for the figure.

5. The contraction factor for an n-gon is denoted by r_n, where

$$r_n = \frac{1}{2\left(1+\sum_{k=1}^{\lfloor n/4 \rfloor}\cos\left(\frac{2k\pi}{n}\right)\right)},$$

and the notation ⌊n/4⌋ refers to the greatest integer not exceeding n/4.

6. Apply translation and rotation so that the baseline becomes the segment joining (0,0) and (1,0).

7. Construct affine mappings using the transformed coordinates and the scaling factor r_n.

8. Form a super IFS from the affine mappings derived in Step 7.

9. Prepare V input panels and V output panels and load an image on each input panel for iteration.

10. Implement Ishikawa iteration on the super IFS obtained in Step 8.

11. Apply the affine maps computed in Step 10 repeatedly (in Fig. 2, n = 17) and allow the system to evolve until the desired disconnected polygonal superfractals appear.

Remark 2. Employing this procedure with n=3 yields the classical Sierpiński triangle, while n = 4 produces the Sierpiński carpet [16].

3.1 5-variable disconnected Sierpinski pentagonal fractals

To develop an IFS for a disconnected pentagon-based superfractal, the horizontal and vertical scaling ratio r = 1/3 is used. Although Algorithm 1 gives the true scaling factor r = 0.381966, here r = 1/3 is deliberately chosen so that the resulting IFS remains disconnected. The derivation of the exact value can be found in [19, 20].

To generate a disconnected regular pentagonal fractal, we define five operator systems, each consisting of a single affine transformation:

$G_1 = \{\square; g_1^1; q_1^1\}$, $G_2 = \{\square; g_1^2; q_1^2\}$,
$G_3 = \{\square; g_1^3; q_1^3\}$, $G_4 = \{\square; g_1^4; q_1^4\}$, $G_5 = \{\square; g_1^5; q_1^5\}$.

Here g_i^j is an affine map and q_i^j is its corresponding probability, given by

$$g_1^1 = (u/3, v/3 + 1/5), \quad q_1^1 = 1/5,$$
$$g_1^2 = (u/3, v/3 + 5/4), \quad q_1^2 = 1/5,$$
$$g_1^3 = (u/3 + 5/3, v/3 + 1), \quad q_1^3 = 1/5,$$
$$g_1^4 = (u/3 + 1, v/3 + 1/2), \quad q_1^4 = 1/5,$$
$$g_1^5 = (u/3 + 5/3, v/3), \quad q_1^5 = 1/5,$$

and

$$\sum q_i^j = 1 \text{ for each j.}$$

Since the selection probability of each operator is equal, five blank output screens $\square'_1, \square'_2, \square'_3, \square'_4, \square'_5$ are prepared.

We begin with five sets of contraction mappings $\{g_1^1\}$, $\{g_1^2\}$, $\{g_1^3\}$, $\{g_1^4\}$, $\{g_1^5\}$, where each $g_m^n: \square \to \square$ is defined on the unit square $[0,1]\times[0,1] \subset \mathbb{R}^2$.

Five collections of screens are used, each representing □. The input screens are

$$(\square_1, \square_2, \square_3, \square_4, \square_5),$$

and the output screens are

$$(\square'_1, \square'_2, \square'_3, \square'_4, \square'_5).$$

Initially, each input screen receives a seed image (see Fig. 2), while outputs are cleared.

To synthesise the pentagonal system as a 5-variable superfractal, Algorithm 2 (Appendix) is employed. See Fig. 2 for the 5-variable disconnected pentagonal output obtained using λ =0.89 and μ = 0.98.

For

$$0.78 \le \lambda, \mu \le 1,$$

the iterates produce a good pentagonal approximation. For 0.78 < λ,μ > 1, convergence deteriorates for these choices of λ and μ.

3.2 3-variable disconnected Sierpin´ski hexagonal fractal

To build an IFS for the hexagonal case, the vertical contraction ratio r_1 = 1/3 and horizontal contraction ratio r_2 = 1/4 are employed.

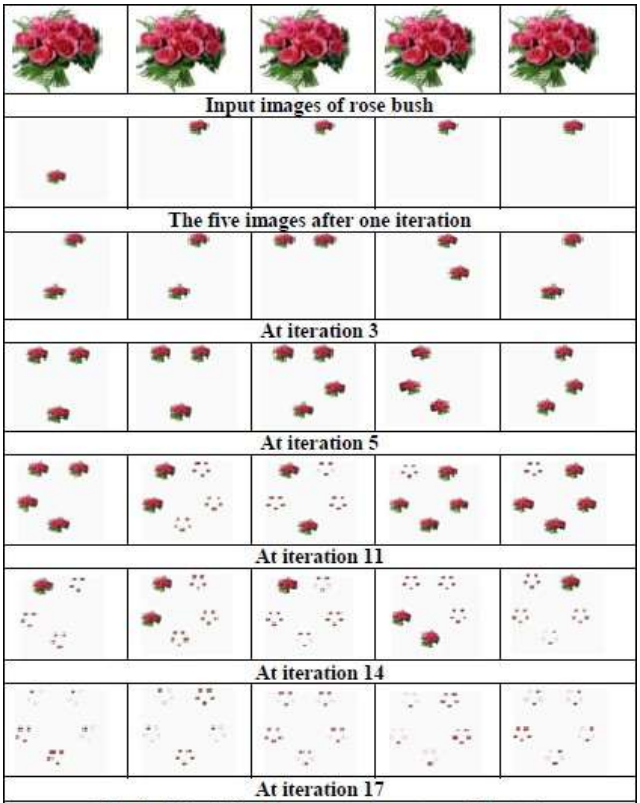

Fig. 2: 5-Variable disconnected pentagonal fractal

The actual scaling constant obtained via Algorithm 1 is r = 1/3; r_2 is chosen smaller to ensure that the resulting IFS is disconnected. More details can be found in [19, 20].

To construct a disconnected regular hexagonal fractal, we consider a triple of operator sets, each consisting of two affine maps:

$G_1 = \{\square;\, g_1^1, g_2^1;\, q_1^1, q_2^1\}$, $G_2 = \{\square;\, g_1^2, g_2^2;\, q_1^2, q_2^2\}$,
$G_3 = \{\square;\, g_1^3, g_2^3;\, q_1^3, q_2^3\}$.

Here, g_i^j denotes an affine transformation and q_i^j gives the associated probability weight, where

$$g_1^1 = (u/3,\, \upsilon/4+1/4), \quad q_1^1 = 1/2,$$
$$g_2^1 = (u/3,\, \upsilon/4+3/4), \quad q_2^1 = 1/2,$$
$$g_1^2 = (u/3+1/2,\, \upsilon/4+1), \quad q_1^2 = 1/2,$$
$$g_2^2 = (u/3+1,\, \upsilon/4+3/4), \quad q_2^2 = 1/2,$$
$$g_1^3 = (u/3+1,\, \upsilon/4+1/4), \quad q_1^3 = 1/2,$$
$$g_2^3 = (u/3+1/2,\, \upsilon/4), \quad q_2^3 = 1/2,$$

and

$$\sum q_i^j = 1 \text{ for each j.}$$

All three operators G_1, G_2 and G_3 are taken with equal selection probability. Three blank output screens $\square'_1$, $\square'_2$, $\square'_3$ are used.

We work with three collections of contractive affine mappings:

$$\{g_1^1, g_2^1\}, \qquad \{g_1^2, g_2^2\}, \qquad \{g_1^3, g_2^3\},$$

where each map $g_m^n : \square \to \square$ acts on the unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$.

Two groups of screens are employed, each screen representing a copy of $\square$:

$$(\square_1, \square_2, \square_3)$$

Serves as the input set, and

$$(\square'_1, \square'_2, \square'_3)$$

functions as the output set.

Initially, each input screen contains a seed image as shown in Fig. 3 (Flying Bird examples), while output screens are cleared.

To synthesise the disconnected hexagon as a 3–variable superfractal, Algorithm 3 is applied (Appendix); see Fig. 3 for the resulting structure for parameter values $\lambda = 0.8$, $\mu = 0.89$. For this regular hexagonal case, convergence of the Ishikawa iteration is observed whenever $\lambda \geq 0.78$ and $\mu \leq 1$.

IV. CONCLUSION

This work provides a general scheme to determine vertex locations and scaling ratios of a regular n-gon and details its construction process. Using this procedure, V-variable disconnected pentagonal and hexagonal superfractals have been generated via the Ishikawa orbit. For the construction of regular pentagonal and hexagonal fractals, the Ishikawa iteration successfully produces accurate approximations for $\lambda \geq 0.78$, $\mu \leq 1$.

The paper offers an alternative perspective for studying n-gon structures through totally disconnected hyperbolic IFS frameworks.

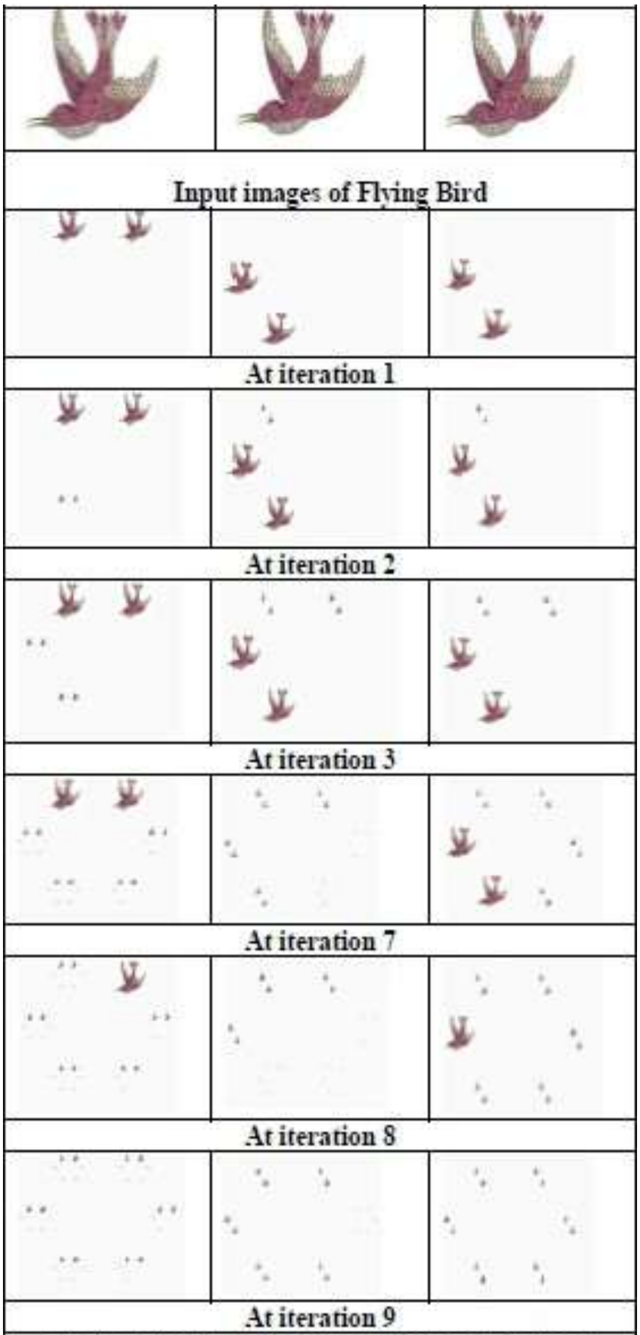

Fig. 3: 3-Variable disconnected hexagonal fractal

Appendix

Algorithm 2

1. Begin with five operators G_1, G_2, G_3, G_4, G_5 five display panels $\square_1$, $\square_2$, $\square_3$, $\square_4$, $\square_5$ and five blank destination panels $\square'_1$, $\square'_2$, $\square'_3$, $\square'_4$, $\square'_5$. An identical image is placed on each source panel, although different initial images could be chosen if desired.

2. Choose randomly one of the operators G_1, G_2, G_3, G_4, G_5, say $\{f_1^{n_1}\}$. Execute the Ishikawa updating rule: $(\lambda f_1^{n_1}(u,v)+(1-\lambda)[\mu f_1^{n_1}(u,v)+(1\text{-}u)u]$, $\lambda f_1^{n_1}(u,v)+(1-\lambda)[\mu f_1^{n_1}(u,v)+(1\text{-}u)v])$ on a randomly selected image from $\square 1, \square 2, \square 3, \square 4, \square 5$, and place the resulting image over $\square'_1$.

3. Again, choose at random an operator, say $\{f_1^{n_2}\}$. Apply $(\lambda f_1^{n_2}(u,v)+(1-\lambda)[\mu f_1^{n_2}(u,v)+(1\text{-}u)u]$, $\lambda f_1^{n_2}(u,v)+(1-\lambda)[\mu f_1^{n_2}(u,v)+(1\text{-}u)v])$ to a randomly chosen image on $\square_1$, $\square_2$, $\square_3$, $\square_4$, $\square_5$and store the output on $\square'_2$.

4. Next, select randomly one of the remaining four operators, say $\{f_1^{n_3}\}$, and apply $(\lambda f_1^{n_3}(u,v)+(1-\lambda)[\mu f_1^{n_3}(u,v)+(1\text{-}u)u]$, $\lambda f_1^{n_3}(u,v)+(1-\lambda)[\mu f_1^{n_3}(u,v)+(1\text{-}u)v])$ on a randomly chosen input image, depositing the processed output onto $\square'_3$.

5. Again, pick a remaining operator at random, say $\{f_1^{n_4}\}$, and evaluate $(\lambda f_1^{n_4}(u,v)+(1-\lambda)[\mu f_1^{n_4}(u,v)+(1\text{-}u)u]$, $\lambda f_1^{n_4}(u,v)+(1-\lambda)[\mu f_1^{n_4}(u,v)+(1\text{-}u)v])$ on one of the images selected randomly; overlay the resulting frame on $\square'_4$.

6. Finally, select the last unchosen operator, say $\{f_1^{n_5}\}$, and compute $(\lambda f_1^{n_5}(u,v)+(1-\lambda)[\mu f_1^{n_5}(u,v)+(1\text{-}u)u]$, $\lambda f_1^{n_5}(u,v)+(1-\lambda)[\mu f_1^{n_5}(u,v)+(1\text{-}u)v])$ from another randomly selected input image, placing the transformed output on $\square'_5$.

7. Treat $\square'_1$, $\square'_2$, $\square'_3$, $\square'_4$, $\square'_5$ as the new collection of input panels.

8. Iterate Steps 2–7 sufficiently many times (say k repetitions) to obtain the required 5-variable disconnected pentagonal superfractal.

Algorithm 3

1. Begin with operators G_1, G_2, G_3, three initial screens $\square_1$, $\square_2$, $\square_3$, and three cleared output screens $\square'_1$, $\square'_2$, $\square'_3$. Each input screen initially displays the same figure, although distinct starting images may also be used.

2. Select at random one operator among G_1, G_2, G_3, say $\{f_1^{n_1}, f_2^{n_1}\}$. Apply $(\lambda f_1^{n_1}(u,v)+(1-\lambda)[\mu f_1^{n_1}(u,v)+(1\text{-}u)u]$, $\lambda f_1^{n_1}(u,v)+(1-\lambda)[\mu f_1^{n_1}(u,v)+(1\text{-}u)v])$ to a randomly chosen input image and place the output on $\square'_1$. Then compute $(\lambda f_1^{n_1}(u,v)+(1-\lambda)[\mu f_1^{n_1}(u,v)+(1\text{-}u)u]$, $\lambda f_1^{n_1}(u,v)+(1-\lambda)[\mu f_1^{n_1}(u,v)+(1\text{-}u)v])$ on another randomly selected input displaying overlaying the result over the current image on $\square'_1$.

3. Again, select randomly one operatorG_1, G_2, G_3, say $\{f_1^{n_2}, f_2^{n_2}\}$. Apply $(\lambda f_1^{n_2}(u,v)+(1-\lambda)[\mu f_1^{n_2}(u,v)+(1\text{-}u)u]$,

$\lambda f_1^{n_2}(u,v)+(1-\lambda)[\mu f_1^{n_2}(u,v)+(1-u)v])$ to a randomly chosen input image and overlay the computed output on $\square'_2$. Now apply $(\lambda f_2^{n_2}(u,v)+(1-\lambda)[\mu f_2^{n_2}(u,v)+(1-u)u]$, $\lambda f_2^{n_2}(u,v)+(1-\lambda)[\mu f_2^{n_2}(u,v)+(1-u)v])$ to another randomly selected image and place its output over the existing one on $\square'_2$.

4. Again, select randomly an operator among G_1, G_2, G_3, say $\{f_1^{n_3}, f_2^{n_3}\}$. Apply $(\lambda f_1^{n_3}(u,v)+(1-\lambda)[\mu f_1^{n_3}(u,v)+(1-u)u]$, $\lambda f_1^{n_3}(u,v)+(1-\lambda)[\mu f_1^{n_3}(u,v)+(1-u)v])$ to a randomly chosen image and place the output on $\square'_3$. Then process $(\lambda f_2^{n_3}(u,v)+(1-\lambda)[\mu f_2^{n_3}(u,v)+(1-u)u]$, $\lambda f_2^{n_3}(u,v)+(1-\lambda)[\mu f_2^{n_3}(u,v)+(1-u)v])$ and overlay this output on top of the existing content on $\square'_3$.

5. Regard $\square'_1$, $\square'_2$, $\square'_3$ as the refreshed set of input screens.

6. Perform Steps 2–5 repeatedly (say k times) until the target 3-variable disconnected hexagonal superfractal emerges.

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